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Synthetic Braided Geometry. I

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In a previous paper we dealt with supergeometry from a synthetic standpoint, showing that the totality of vector fields on a superized version of microlinear space is a Lie superalgebra. The main purpose of this paper is to generalize the methods to symmetric braided geometry. Nonsymmetric braided geometry will be discussed in a sequel to this paper.

0. INTRODUCTION

Synthetic differential geometry provides a natural framework for differential geometry in which not only global and local, but also infinitesimal horizons are existent and emphasized. It goes without saying that standard differential geometry is the study of differential manifolds, which are defined to be spaces diffeomorphic *locally* to Euclidean spaces. Synthetic differential geometry is the study of microlinear spaces, which are defined to be spaces *infinitesimally* indistinguishable from Euclidean spaces. Such locutions as "vector fields are infinitesimal transformations" are only rhetorical in standard differential geometry, but essential in synthetic differential geometry. Synthetic differential geometry is by no means a trifling reformulation of standard differential geometry in infinitesimal terms. That the totality of vector fields on a differential manifold is a Lie algebra is a truism in standard differential geometry because of the coincidence of vector fields on a differential manifold with derivations on its function algebra, but its synthetic equivalent that the totality of vector fields on a microlinear space is a Lie algebra occupies a naggingly ticklish position in synthetic differential geometry. For a good introduction to synthetic differential geometry the reader is referred to Lavendhomme (1996).

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Supergeometry is a study of supermanifolds, which are a generalization of differential manifolds so as to include fermionic aspects besides bosonic ones. Fermionic entities are infinitesimal in essence, for their squares always vanish. Therefore supergeometry is an infinitesimal generalization of standard differential geometry. Supergeometry lies at the entrance to noncommutative geometry in the sense that the set of real supernumbers is not commutative, but graded-commutative. For a good introduction to supergeometry the reader is referred to Manin (1988).

Braided geometry is an elegant and far-reaching generalization of supergeometry, in which the category of vector spaces is replaced by a braided monoidal category. It has been pioneered and championed by Majid (1995a,b), Marcinek (1994), and others. The standard gadget for transmogrifying braided geometry into noncommutative geometry is bosonization, while the standard device for translating noncommutative geometry into braided geometry is transmutation. If the braiding is symmetric, braided geometry lies at the very periphery of supergeometry, but encompasses not only supergeometry (based on Bose–Fermi statistics), but also geometries based on such exotic statistics as anyonic or color ones.

Synthetic treatments of supergeometry have been discussed by Nishimura (1998b) and Yetter (1988). The principal objective of this paper is to present a synthetic treatment of symmetric braided geometry along the lines of the former. Nonsymmetric braided geometry will be discussed synthetically in a sequel to this paper. We assume that the reader is familiar with Lavendhomme's (1996) monograph on synthetic differential geometry up to Chapter 3. As is usual in synthetic differential geometry, the reader should presume that we are working in a non-Boolean topos, so that the principle of excluded middle and Zorn's lemma should be avoided. But for these two points, we could feel that we are working in the standard universe of sets.

1. BASIC BRAIDED ALGEBRA

We choose, once and for all, a braided monoidal category $\mathfrak{C} = (\mathscr{C}, \otimes, \mathbf{1}, \Phi, l, r, \Psi)$ satisfying the following conditions:

(1.1) $\mathscr{C}$ is a subcategory of the category of all $\mathtt{k}$-linear spaces with a field $\mathtt{k}$.
(1.2) $\otimes$ is the standard tensor product of $\mathtt{k}$-linear spaces.
(1.3) The unit object $\mathbf{1}$ is $\mathtt{k}$ regarded as a $\mathtt{k}$-linear space in the standard manner.

(1.4) The associativity constraint Φ, the left unit constraint l, and the right unit constraint r are the standard ones of $\Bbbk$-linear spaces.
(1.5) The braiding Ψ is symmetric in the sense that $\Psi_{W,V} \circ \Psi_{V,W} = 1_{V \otimes W}$ for any objects V, W in $\mathscr{C}$.
(1.6) There exists a finite set Π of mutually nonisomorphic objects of $\mathscr{C}$ including the unit object $\mathbf{1}$, say, $\Pi = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \ldots, \mathbf{k}\}$, such that:
(1.6.1) Every object $\mathbf{p}$ in Π is a one-dimensional $\Bbbk$-linear space.
(1.6.2) The set Π is closed under $\otimes$, i.e., for any objects $\mathbf{p}$, $\mathbf{q}$ in Π, there exists an object $\mathbf{r}$ in Π such that $\mathbf{p} \otimes \mathbf{q}$ is isomorphic to $\mathbf{r}$ in the category $\mathscr{C}$ (we will use $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$, ... with or without subscripts as variables over Π).
(1.6.3) Every direct sum of (possibly infinitely many) copies of objects in Π as well as all its associated canonical injections and projections belong to $\mathscr{C}$, and any object in $\mathscr{C}$ is a direct sum of copies of objects in Π.
(1.7) For any morphism $\alpha: U \to V$ in $\mathscr{C}$, if α happens to be an isomorphism of $\Bbbk$-linear spaces, then $\alpha^{-1}: V \to U$ belongs to $\mathscr{C}$, so that α is an isomorphism in $\mathscr{C}$.

As is the custom in dealing with monoidal categories, we will often proceed as if the monoidal category $(\mathscr{C}, \otimes, \mathbf{1}, \Phi, l, r)$ were strict, which is justifiable by Theorem XI.5.3 of Kassel (1995). We will often write $\mathbf{p} + \mathbf{q}$ for r isomorphic to $\mathbf{p} \otimes \mathbf{q}$ in (1.6.2). Then it is easy to see the following:

Proposition 1.1. Π is an abelian monoid with respect to the operation $+$ defined above.

Proof. The associativity constraint $\Phi_{\mathbf{p},\mathbf{q},\mathbf{r}}: (\mathbf{p} \otimes \mathbf{q}) \otimes \mathbf{r} \to \mathbf{p} \otimes (\mathbf{q} \otimes \mathbf{r})$ guarantees that Π is a semigroup. The left unit constraint $l_{\mathbf{p}}: \mathbf{1} \otimes \mathbf{p} \to \mathbf{p}$ and the right unit constraint $r_{\mathbf{p}}: \mathbf{p} \otimes \mathbf{1} \to \mathrm{p}$ warrant that Π is not only a semigroup, but a monoid. The commutativity of the monoid Π follows from the braiding $\Psi_{\mathbf{p},\mathbf{q}}: \mathbf{p} \otimes \mathbf{q} \to \mathbf{q} \otimes \mathbf{p}$. ∎

We choose an arbitrary nonzero element $x_{\mathbf{p}}$ of each one-dimensional $\Bbbk$-linear space $\mathbf{p}$ in Π once and for all. For $\mathbf{p}$, $\mathbf{q}$ in Π there exists a unique $\delta^{\mathbf{p},\mathbf{q}} \in \Bbbk$ such that

(1.8) $\Psi_{\mathbf{p},\mathbf{q}}(x_{\mathbf{p}} \otimes x_{\mathbf{q}}) = \delta^{\mathbf{p},\mathbf{q}}(x_{\mathbf{q}} \otimes x_{\mathbf{p}})$

It is easy to see that the numbers $\delta^{\mathbf{p},\mathbf{q}}$ do not depend on our particular choice $\{x_{\mathbf{p}}\}_{\mathbf{p} \in \Pi}$.

Proposition 1.2. The numbers $\delta^{\mathbf{p},\mathbf{q}}$ satisfy the following identities:

(1.9) $\delta^{\mathbf{p},\mathbf{q}}\delta^{\mathbf{q},\mathbf{p}} = 1$
(1.10) $\delta^{\mathbf{p},\mathbf{q}+\mathbf{r}} = \delta^{\mathbf{p},\mathbf{q}}\delta^{\mathbf{p},\mathbf{r}}$
(1.11) $\delta^{\mathbf{p}+\mathbf{q},\mathbf{r}} = \delta^{\mathbf{p},\mathbf{r}}\delta^{\mathbf{q},\mathbf{r}}$

Proof. (1.9) follows from the assumption (1.5) that $\Psi_{\mathbf{p},\mathbf{q}} \circ \Psi_{\mathbf{q},\mathbf{p}} = 1_{\mathbf{p}\otimes\mathbf{q}}$. (1.10) and (1.11) follow from the so-called hexagon axiom, which claims that $\Psi_{\mathbf{p},\mathbf{q}\otimes\mathbf{r}} = (1_{\mathbf{q}} \otimes \Psi_{\mathbf{p},\mathbf{r}}) \circ (\Psi_{\mathbf{p},\mathbf{q}} \otimes 1_{\mathbf{r}})$ and $\Psi_{\mathbf{p}\otimes\mathbf{q},\mathbf{r}} = (\Psi_{\mathbf{p},\mathbf{r}} \otimes 1_{\mathbf{q}}) \circ (1_{\mathbf{p}} \otimes \Psi_{\mathbf{q},\mathbf{r}})$ up to associativity and unit constraints. ■

If Π happens to be a group, then the pair (Π, δ) is a signed group in terms of Marcinek (1991).

Given an object U in $\mathscr{C}$, the direct sum decomposition of U into objects in Π in (1.6.3) is not unique, but the **p**-*component* of U defined as the direct sum of the images of all the canonical injections from **p** into U with respect to a particular decomposition of U will soon turn out to be independent of our choice of a particular decomposition of U. Therefore we can safely write $U_{\mathbf{p}}$ for the **p**-component of U.

Proposition 1.3. Let Ξ and Ξ' be two direct sum decompositions of U in (1.6.3). Then, for any p in Π, the p-components $U_{\mathbf{p}}^{\Xi}$ and $U_{\mathbf{p}}^{\Xi'}$ of U with respect to Ξ and Ξ' coincide.

Proof. The proof uses a gimmick which is familiar in the proof of the well-known fact of algebra that, although a direct sum decomposition of a semisimple module into simple ones is not unique, its homogeneous component affiliated to a particular simple module is well defined, for which the reader is referred, e.g., to Wisbauer (1991, Chapter 4). For any canonical injection ι of **p** into U in the decomposition Ξ and any canonical projection π of U onto **q** in the decomposition Ξ' with $\mathbf{p} \neq \mathbf{q}$, $\pi \circ \iota = 0$, for otherwise **p** and **q** would be isomorphic in $\mathscr{C}$ by (1.7). This means that $U_{\mathbf{p}}^{\Xi} \subset U_{\mathbf{p}}^{\Xi'}$ for any **p** in Π. By interchanging the roles of Ξ and Ξ' in the above discussion, we have that $U_{\mathbf{p}}^{\Xi'} \subset U_{\mathbf{p}}^{\Xi}$ for any **p** in Π. Therefore the desired conclusion follows. ■

Corollary 1.4. $U = U_1 \oplus \cdots \oplus U_k$, so that each $u \in U$ can be decomposed uniquely as $u = u_1 + \cdots + u_k$ with $u_{\mathbf{p}} \in U_{\mathbf{p}}$ for any **p** in Π. ■

An element u of U which happens to consist in $U_{\mathbf{p}}$ for some **p** in Π is called *pure* (*of grade* p), in which we will denote **p** by $|u|$.

The same gadget used in the proof of Proposition 1.3 establishes the following:

Proposition 1.5. Any morphism α: $U \to V$ in $\mathscr{C}$ preserves grading [i.e., $\alpha(U_{\mathbf{p}}) \subset V_{\mathbf{p}}$ for each p in Π]. ■

We now enjoin that the class of morphisms in $\mathcal{C}$ be saturated with respect to this property in the following sense:

(1.12) For any objects U, V in $\mathcal{C}$, if a homomorphism $\alpha\colon U \to V$ of $\Bbbk$-linear spaces preserves grading [i.e., $\alpha(U_{\mathbf{p}}) \subset V_{\mathbf{p}}$ for any $\mathbf{p}$ in Π], then α lies in $\mathcal{C}$.

The notion of an algebra in the braided monoidal category $\mathfrak{C}$, usually called a $\mathfrak{C}$-*algebra*, can be defined diagrammatically as in Kassel (1995, §III.1). A $\mathfrak{C}$-algebra $\mathscr{A}$ with its product $\mu_{\mathscr{A}}\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ is said to be $\mathfrak{C}$-*commutative* if $\mu_{\mathscr{A}} \circ \Psi_{\mathscr{A},\mathscr{A}} = \mu_{\mathscr{A}}$. Given a $\mathfrak{C}$-algebra $\mathscr{A}$, the notions of a left $\mathscr{A}$-module and a right $\mathscr{A}$-module in $\mathfrak{C}$, usually called a *left* $\mathscr{A}$-$\mathfrak{C}$-*module* and a *right* $\mathscr{A}$-$\mathfrak{C}$-*module*, respectively, can be defined diagrammatically as in Majid (1995a, §1.6). If $\mathscr{A}$ happens to be $\mathfrak{C}$-commutative, a left $\mathscr{A}$-$\mathfrak{C}$-module $\mathcal{M}$ with its left action $\eta\colon \mathscr{A} \otimes \mathcal{M} \to \mathcal{M}$ can naturally be converted into a right $\mathscr{A}$-$\mathfrak{C}$-module with its right action $\eta \circ \Psi_{\mathcal{M},\mathscr{A}}\colon \mathcal{M} \otimes \mathscr{A} \to \mathcal{M}$, and vice versa, so that the distinction between left and right is not essential in the $\mathfrak{C}$-commutative case. A left (right, resp.) $\mathscr{A}$-$\mathfrak{C}$-module $\mathcal{M}$ is said to be $\mathfrak{C}$-*finite-dimensional* if there exists a finite-dimensional $\Bbbk$-linear space V in $\mathcal{C}$ such that $\mathscr{A} \otimes V$ ($V \otimes \mathscr{A}$, resp.) is isomorphic to $\mathcal{M}$ as left (right, resp.) $\mathscr{A}$-$\mathfrak{C}$-modules. The notions of a left $\mathscr{A}$-module algebra and a right $\mathscr{A}$-module algebra in $\mathfrak{C}$, usually called a *left* $\mathscr{A}$-$\mathfrak{C}$-*algebra* and a *right* $\mathscr{A}$-$\mathfrak{C}$-*algebra*, respectively, can also be defined diagrammatically as in Majid (1995a, §1.6). An ideal of a $\mathfrak{C}$-algebra $\mathscr{A}$ is said to be a $\mathfrak{C}$-*ideal* if it belongs to $\mathcal{C}$. A $\mathfrak{C}$-commutative $\mathfrak{C}$-algebra $\mathscr{A}$ is called $\mathfrak{C}$-*local* if it has a maximal $\mathfrak{C}$-ideal. Other standard notions such as that of a *homomorphism of* $\mathfrak{C}$-*algebras,* which can easily be formulated diagrammatically, will be used freely. Given a $\mathfrak{C}$-commutative $\mathfrak{C}$-algebra $\mathscr{A}$ and an $\mathscr{A}$-$\mathfrak{C}$-algebra $\mathscr{B}$, $\mathrm{Spec}_{\mathscr{A}}\mathscr{B}$ denotes the totality of homomorphisms of $\mathscr{A}$-$\mathfrak{C}$-algebras from $\mathscr{B}$ into $\mathscr{A}$.

Now we choose, once and for all, a $\mathfrak{C}$-commutative $\mathfrak{C}$-algebra $\mathbb{R}$ intended to play the role of real numbers in our braided mathematics. So we must enjoin the following axiom on $\mathbb{R}$:

(1.13) $\mathbb{R}$ is a $\mathfrak{C}$-commutative $\mathfrak{C}$-algebra.

Another important axiom on $\mathbb{R}$ will be presented in the next section. Given a set Z, the totality of functions from Z to $\mathbb{R}$ is an $\mathbb{R}$-$\mathfrak{C}$-algebra with componentwise operations whose p-component can naturally be identified with the totality of functions from Z to $\mathbb{R}_{\mathbf{p}}$.

Given a finite sequence $\mathbf{p}_1, \ldots, \mathbf{p}_n$ in Π, we can form the tensor $\mathfrak{C}$-algebra $T(\mathbf{p}_1 \oplus \cdots \oplus \mathbf{p}_n)$ of the $\Bbbk$-linear space $\mathbf{p}_1 \oplus \cdots \oplus \mathbf{p}_n$. The quotient $\mathfrak{C}$-algebra of $T(\mathbf{p}_1 \oplus \cdots \oplus \mathbf{p}_n)$ with respect to the $\mathfrak{C}$-ideal generated by $\{x_{\mathbf{p}_j}x_{\mathbf{p}_i} - \delta^{\mathbf{p}_i,\mathbf{p}_j}x_{\mathbf{p}_i}x_{\mathbf{p}_j} \,|\, 1 \le i\ j \le n\}$ is a $\mathfrak{C}$-algebra called the *polynomial* $\mathfrak{C}$-*algebra of variables* $x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}$ and denoted by $\Bbbk[x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}]$. The $\mathbb{R}$-$\mathfrak{C}$-

algebra $\mathbb{R} \otimes \Bbbk[x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}]$ is called the *polynomial* $\mathfrak{C}$*-algebra of variables* $x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}$ *over* $\mathbb{R}$ or the *polynomial* $\mathbb{R}$-$\mathfrak{C}$*-algebra of variables* $x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}$ and is denoted by $\mathbb{R}[x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}]$. The $\mathbb{R}$-$\mathfrak{C}$-algebra $\mathbb{R}[x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}]$ is characterized by the following universality property:

Proposition 1.6. The $\mathbb{R}$-$\mathfrak{C}$-algebra $\mathbb{R}[x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}]$ is $\mathfrak{C}$-commutative. For any $\mathfrak{C}$-commutative $\mathbb{R}$-$\mathfrak{C}$-algebra $\mathcal{A}$ and any morphisms α_i: $\mathbf{p}_i \to \mathcal{A}$ in $\mathcal{C}$ $(1 \leq i \leq n)$, there exists a unique homomorphism α of $\mathbb{R}$-$\mathfrak{C}$-algebras from $\mathbb{R}[x_{\mathbf{p}_1}, \ldots, x_{\mathbf{p}_n}]$ to $\mathcal{A}$ whose restriction to $\mathbf{p}_i$ is α_i $(1 \leq i \leq n)$. ■

A *Lie* $\mathfrak{C}$*-algebra over* $\mathbb{R}$ or a *lie* $\mathbb{R}$-$\mathfrak{C}$*-algebra* is an $\mathbb{R}$-$\mathfrak{C}$-module L with its left $\mathbb{R}$-$\mathfrak{C}$-module structure η: $\mathbb{R} \otimes L \to L$ and its associated right $\mathbb{R}$-$\mathfrak{C}$-module structure η': $L \otimes \mathbb{R} \to L$ which is endowed with a morphism $\mathcal{L}$: $L \otimes L \to L$ in $\mathcal{C}$ satisfying the following conditions:

(1.14) $\mathcal{L} \circ \eta_{12} = \eta \circ \mathcal{L}_{23}$ on $\mathbb{R} \times L \times L$
(1.15) $\mathcal{L} \circ \eta'_{23} = \eta' \circ \mathcal{L}_{12}$ on $L \times L \times \mathbb{R}$
(1.16) $\mathcal{L} \circ \Psi = -\mathcal{L}$ on $L \times L$
(1.17) $\mathcal{L} \circ \mathcal{L}_{23} + \mathcal{L} \circ \mathcal{L}_{23} \circ \Psi_{23} \circ \Psi_{12} + \mathcal{L} \circ \mathcal{L}_{23} \circ \Psi_{12} \circ \Psi_{23} = 0$ on $L \times L \times L$

In the above list of conditions such notations as $\mathcal{L}_{23}$ are the familiar conventions in the realm of quantum groups, for which the reader is referred to Kassel (1995, §VIII.2). Given u, $v \in L$, we will often write $[u, v]$ for $\mathcal{L}(u \otimes v)$. Conditions (1.16) and (1.17) can be rephrased in the following form:

Proposition 1.7. Conditions (1.16) and (1.17) are equivalent to the following conditions, respectively:

(1.18) $[v, u] = -\delta^{\mathbf{q},\mathbf{p}}[u, v]$ for any $u \in L_{\mathbf{p}}$ and any $v \in L_{\mathbf{q}}$.
(1.19) $[u, [v, w]] + \delta^{\mathbf{p},\mathbf{q}+\mathbf{r}}[v, [w, u]] + \delta^{\mathbf{p}+\mathbf{q},\mathbf{r}}[w, [u, v]] = 0$ for any $u \in L_{\mathbf{p}}$, any $v \in L_{\mathbf{q}}$, and any $w \in L_{\mathbf{r}}$. ■

2. WEIL $\mathfrak{C}$-ALGEBRAS AND $\mathfrak{C}$-MICROLINEARITY

A *Weil* $\mathfrak{C}$*-algebra* is a $\mathfrak{C}$-local $\mathfrak{C}$-commutative $\mathbb{R}$-$\mathfrak{C}$-algebra $\mathcal{M}$ with an $\mathbb{R}$-$\mathfrak{C}$-finite-dimensional maximal $\mathfrak{C}$-ideal $\mathfrak{m}$ for which $\mathcal{M} = \mathbb{R} \oplus \mathfrak{m}$ (the first component is the $\mathbb{R}$-$\mathfrak{C}$-algebra structure). By way of example, the quotient $\mathfrak{C}$-algebra of the polynomial $\mathfrak{C}$-algebra $\mathbb{R}[x_1, \ldots, x_n]$ with respect to the $\mathfrak{C}$-ideal generated by $\{x_i x_j | 1 \leq i \leq n\}$ is a Weil $\mathfrak{C}$-algebra and is denoted by $\mathcal{M}(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ with $\mathbf{p}_i = |x_i|$ $(1 \leq i \leq n)$. Given Weil $\mathfrak{C}$-algebras $\mathcal{M}_1$ and $\mathcal{M}_2$ with maximal $\mathfrak{C}$-ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively, a homomorphism of $\mathbb{R}$-$\mathfrak{C}$-algebras φ: $\mathcal{M}_1 \to \mathcal{M}_2$ is said to be a *homomorphism of Weil* $\mathfrak{C}$*-algebras* if it preserves maximal $\mathfrak{C}$-ideals, i.e., if $\varphi(\mathfrak{m}_1) \subset \mathfrak{m}_2$. A finite limit diagram of $\mathbb{R}$-$\mathfrak{C}$-algebras is said to be a *good finite limit diagram of Weil* $\mathfrak{C}$*-algebras* if every object occurring in the diagram is a Weil $\mathfrak{C}$-algebra and every morphism occurring in the diagram is a homomorphism of Weil $\mathfrak{C}$-algebras. The diagram obtained from a good finite limit diagram of Weil $\mathfrak{C}$-algebras by taking $\mathrm{Spec}_{\mathbb{R}}$ is called a *quasi-colimit diagram of* $\mathfrak{C}$*-small objects.*

The braided version of the general Kock axiom, called the *general* $\mathfrak{C}$*-Kock axiom,* goes as follows:

(2.1) For any Weil $\mathfrak{C}$-algebra $\mathcal{M}$, the canonical $\mathbb{R}$-$\mathfrak{C}$-algebra homomorphism $\mathcal{M} \to \mathbb{R}^{\mathrm{Spec}_{\mathbb{R}}(\mathcal{M})}$ is an isomorphism.

Spaces of the form $\mathrm{Spec}_{\mathbb{R}}(\mathcal{M})$ for some Weil $\mathfrak{C}$-algebras $\mathcal{M}$ are called $\mathfrak{C}$*-infinitesimal spaces* or $\mathfrak{C}$*-small objects*. The $\mathfrak{C}$-infinitesimal space corresponding to Weil $\mathfrak{C}$-algebra $\mathcal{M}(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ is denoted by $D(\mathbf{p}_1, \ldots, \mathbf{p}_n)$. In particular, D corresponding to Weil $\mathfrak{C}$-algebra $\mathbb{R}$ is denoted by 1. The mapping from 1 to a $\mathfrak{C}$-infinitesimal space $\mathrm{Spec}_{\mathbb{R}}(\mathcal{M})$ corresponding to the canonical projection $\mathcal{M} \to \mathbb{R}$ is usually denoted by 0.

The $\mathfrak{C}$-infinitesimal space $D(\mathbf{1}, \ldots, \mathbf{k})$ will play a very important role in our discussion of tangency. First we note that $D(\mathbf{1}, \ldots, \mathbf{k})$ can be identified with the subset of $\mathbb{R}$ consisting of all $d \in \mathbb{R}$ such that $d_{\mathrm{p}}d_{\mathrm{q}} = 0$ for any $\mathbf{p}$, $\mathbf{q} \in \Pi$. Under this identification $(d_1, \ldots, d_{\mathbf{k}}) \in D(\mathbf{1}, \ldots, \mathbf{k})$ corresponds to $d_1 + \cdots + d_{\mathbf{k}} \in \mathbb{R}$. What concerns us most about $D(\mathbf{1}, \ldots, \mathbf{k})$ is that it is, regarded as a subset of $\mathbb{R}$, closed under the left and right actions of $\mathbb{R}$ on itself. More specifically, given $a \in \mathbb{R}$ and $(d_1, \ldots, d_{\mathbf{k}}) \in D(\mathbf{1}, \ldots, \mathbf{k})$, $a(d_1, \ldots, d_{\mathbf{k}})$ and $(d_1, \ldots, d_{\mathbf{k}})a$ are $(e_1, \ldots, e_{\mathbf{k}})$ and $(f_1, \ldots, f_{\mathbf{k}})$ respectively, where $e_{\mathbf{p}}$ is the sum of $a_{\mathbf{q}}d_{\mathbf{r}}$'s and $f_{\mathbf{p}}$ is that of $d_{\mathbf{q}}a_{\mathbf{r}}$'s with $\mathbf{q} + \mathbf{r} = \mathbf{p}$.

Just as the general Kock axiom paved the way to the introduction of a microlinear space, its braided version invokes the notion of a $\mathfrak{C}$*-microlinear space*, which is by definition a space $\mathcal{M}$ satisfying the following condition:

(2.2) For any good finite limit diagram of Weil $\mathfrak{C}$-algebras with its limit $\mathcal{M}$, the diagram obtained by taking $\mathrm{Spec}_{\mathbb{R}}$ and then exponentiating over $\mathcal{M}$ is a limit diagram with its limit $\mathcal{M}^{\mathrm{Spec}_{\mathbb{R}}\mathcal{M}}$.

The following proposition guarantees that we have many $\mathfrak{C}$-microlinear spaces.

Proposition 2.1. (1) $\mathbb{R}_{\mathbf{p}}$ is a $\mathfrak{C}$-microlinear space for any $\mathbf{p}$ in Π.

(2) The class of $\mathfrak{C}$-microlinear spaces is closed under limits and exponentiation by an arbitrary space.

Proof. Statement (1) follows directly from axiom (2.1), while statement (2) can be established as in Lavendhomme (1996, §§2.3, Proposition 1). ■

Proposition 2.2. The diagram

$$\begin{array}{ccccc} & & D(\mathbf{p}) & & \\ & \swarrow & \downarrow \mathtt{j}_1^{\mathbf{p},\mathbf{q}} & & \\ 1 & \xrightarrow{0} & D(\mathbf{p})\times D(\mathbf{q}) & \xrightarrow{\mathtt{m}^{\mathbf{p},\mathbf{q}}} & D(\mathbf{p}+\mathbf{q}) \\ & \nwarrow & \uparrow \mathtt{j}_2^{\mathbf{p},\mathbf{q}} & & \\ & & D(\mathbf{q}) & & \end{array}$$

is a quasi-colimit diagram of $\mathfrak{C}$-small objects, where

(2.3) $\mathtt{j}_1^{\mathbf{p},\mathbf{q}}(d) = (d, 0)$ for any $d \in D(\mathbf{p})$
(2.4) $\mathtt{j}_2^{\mathbf{p},\mathbf{q}}(d) = (0, d)$ for any $d \in D(\mathbf{q})$
(2.5) $\mathtt{m}^{\mathbf{p},\mathbf{q}}(d_1, d_2) = d_1 d_2$ for any $(d_1, d_2) \in D(\mathbf{p}) \times D(\mathbf{q})$

Proof. As in Lavendhomme (1996, §2.2, Proposition 7). ■

Corollary 2.3. Let $\mathcal{M}$ be a $\mathfrak{C}$-microlinear space and $m \in \mathcal{M}$. Let γ be a function from $D(\mathbf{p}) \times D(\mathbf{q})$ to $\mathcal{M}$ such that $\gamma(d_1, 0) = \gamma(0, d_2) = m$ for any $d_1 \in D(\mathbf{p})$ and any $d_2 \in D(\mathbf{q})$. Then there exists a unique function θ: $D(\mathbf{p} + \mathbf{q}) \to \mathcal{M}$ such that $\gamma(d_1, d_2) = \theta(d_1 d_2)$ for any $(d_1, d_2) \in D(\mathbf{p}) \times D(\mathbf{q})$.

Proposition 2.6. The diagrams

$$\begin{array}{ccc} 1 & \xrightarrow{0} & D(\mathbf{q}) \\ {\scriptstyle 0}\downarrow & & \downarrow {\scriptstyle i_2^{\mathbf{p},\mathbf{q}}} \\ D(\mathbf{p}) & \xrightarrow[i_1^{\mathbf{p},\mathbf{q}}]{} & D(\mathbf{p}, \mathbf{q}) \end{array}$$

$$\begin{array}{ccc} 1 & \xrightarrow{0} & D(\mathbf{1}, \ldots, \mathbf{k}) \\ {\scriptstyle 0}\downarrow & & \downarrow {\scriptstyle i_2^{(\mathbf{1},\ldots,\mathbf{k})^2}} \\ D(\mathbf{1}, \ldots, \mathbf{k}) & \xrightarrow[i_1^{(\mathbf{1},\ldots,\mathbf{k})^2}]{} & D(\mathbf{1}, \ldots, \mathbf{k}, \mathbf{1}, \ldots, \mathbf{k}) \end{array}$$

are quasi-colimit diagrams of $\mathfrak{C}$-small objects, where

(2.6) $i_1^{\mathbf{p},\mathbf{q}}(d) = (d, 0)$ for any $d \in D(\mathbf{p})$
(2.7) $i_2^{\mathbf{p},\mathbf{q}}(d) = (0, d)$ for any $d \in D(\mathbf{q})$
(2.8) $i_1^{(\mathbf{1},\ldots,\mathbf{k})^2}(d_\mathbf{1}, \ldots, d_\mathbf{k}) = (d_\mathbf{1}, \ldots, d_\mathbf{k}, 0, \ldots, 0)$ for any $(d_\mathbf{1}, \ldots, d_\mathbf{k}) \in D(\mathbf{1}, \ldots, \mathbf{k})$
(2.9) $i_2^{(\mathbf{1},\ldots,\mathbf{k})^2}(d_\mathbf{1}, \ldots, d_\mathbf{k}) = (0, \ldots, 0, d_\mathbf{1}, \ldots, d_\mathbf{k})$ for any $(d_\mathbf{1}, \ldots, d_\mathbf{k}) \in D(\mathbf{1}, \ldots, \mathbf{k})$

Proof. As in Lavendhomme (1996, §2.2, Proposition 6). ■

Corollary 2.5. Let $\mathcal{M}$ be a $\mathfrak{C}$-microlinear space and $\mathrm{m} \in \mathcal{M}$. For any functions $\gamma_1: D(\mathbf{p}) \to \mathcal{M}$ and $\gamma_2: D(\mathbf{q}) \to \mathcal{M}$ with $\gamma_1(0) = \gamma_2(0) = \mathrm{m}$, there exists a unique function $l^{\mathbf{p},\mathbf{q}}_{(\gamma_1,\gamma_2)}: D(\mathbf{p}, \mathbf{q}) \to \mathcal{M}$ such that $l^{\mathbf{p},\mathbf{q}}_{(\gamma_1,\gamma_2)} \circ i^{\mathbf{p},\mathbf{q}}_1 = \gamma_1$ and $l^{\mathbf{p},\mathbf{q}}_{(\gamma_1,\gamma_2)} \circ i^{\mathbf{p},\mathbf{q}}_2 = \gamma_2$. For any functions $\theta_1, \theta_2: D(\mathbf{1}, \ldots, \mathbf{k}) \to \mathcal{M}$ with $\theta_1(0, \ldots, 0) = \theta_2(0, \ldots, 0) = \mathrm{m}$, there exists a unique function $l^{(\mathbf{1},\ldots,\mathbf{k})^2}_{(\delta_1,\delta_2)}: D(\mathbf{1}, \ldots, \mathbf{k}, \mathbf{1}, \ldots, \mathbf{k}) \to \mathcal{M}$ such that $l^{(\mathbf{1},\ldots,\mathbf{k})^2}_{(\theta_1,\theta_2)} \circ i^{(\mathbf{1},\ldots,\mathbf{k})^2}_1 = \theta_1$ and $l^{(\mathbf{1},\ldots,\mathbf{k})^2}_{(\theta_1,\theta_2)} \circ i^{(\mathbf{1},\ldots,\mathbf{k})^2}_2 = \theta_2$.

3. DIFFERENTIAL CALCULUS

The braided version of the Kock–Lawvere axiom, which is subsumed under the braided version of the general Kock axiom discussed in the previous section, goes as follows:

(3.1) For any function $f: D(\mathbf{p}) \to \mathbb{R}$, there exists a unique $b \in \mathbb{R}$ such that $f(d) = f(0) + bd$ for any $d \in D$.

It is easy to see that this axiom is equivalent to the following:

(3.2) For any function $f: D \to \mathbb{R}$, there exists a unique $b' \in \mathbb{R}$ such that $f(d) = f(0) + db'$ for any $d \in D$.

Indeed, it is easy to see that b in (3.1) and b' in (3.2) determine each other by the following simple relation:

(3.3) $b'_{\mathbf{q}} = \delta^{\mathbf{q},\mathbf{p}} b_{\mathbf{q}}$ for any q in Π.

These two equivalent axioms as a whole are called the $\mathfrak{C}$-*Kock–Lawvere axiom.* The main objective of this section is to discuss some consequences of this axiom without assuming the general $\mathfrak{C}$-Kock axiom.

Given a function $f: \mathbb{R}_{\mathbf{p}} \to \mathbb{R}$ and $a \in \mathbb{R}_{\mathbf{p}}$, by one of the equivalent axioms (3.1) and (3.2), there exist unique $(\overline{\mathbf{D}}_{\mathbf{p}} f)(a) \in \mathbb{R}$ and unique $(f\overleftarrow{\mathbf{D}}_{\mathbf{p}})(a) \in \mathbb{R}$ such that for any $d \in D(\mathbf{p})$,

(3.4) $f(a + d) = f(a) + d(\overline{\mathbf{D}}_{\mathbf{p}} f)(a)$
(3.5) $f(a + d) = f(a) + (f\overleftarrow{\mathbf{D}}_{\mathbf{p}})(a)d$

The functions $a \in \mathbb{R}_{\mathbf{p}} \mapsto (\overline{\mathbf{D}}_{\mathbf{p}} f)(a)$ and $a \in \mathbb{R}_{\mathbf{p}} \mapsto (\overleftarrow{\mathbf{D}}_{\mathbf{p}} f)(a)$ are denoted by $\overline{\mathbf{D}}_{\mathbf{p}} f$ and $f\overleftarrow{\mathbf{D}}_{\mathbf{p}}$, respectively.

Proposition 3.1. Let f and g be functions from $\mathbb{R}_{\mathrm{p}}$ to $\mathbb{R}$. Let $a \in \mathbb{R}$. Then we have

(3.6) $\overline{\mathbf{D}}_{\mathbf{p}}(f + g) = \overline{\mathbf{D}}_{\mathbf{p}} f + \overline{\mathbf{D}}_{\mathbf{p}} g$
(3.7) $(f + g)\overleftarrow{\mathbf{D}}_{\mathbf{p}} = f\overleftarrow{\mathbf{D}}_{\mathbf{p}} + g\overleftarrow{\mathbf{D}}_{\mathbf{p}}$
(3.8) $(af)\overleftarrow{\mathbf{D}}_{\mathbf{p}} = a(f\overleftarrow{\mathbf{D}}_{\mathbf{p}})$

(3.9) $\overline{\underline{\mathbf{D}}}_{\mathbf{p}}(fa) = (\overline{\underline{\mathbf{D}}}_{\mathbf{p}} f)a$

(3.10) $\overline{\mathbf{D}}_{\mathbf{p}}(fg) = (\overline{\mathbf{D}}_{\mathbf{p}} f)g + \delta^{\mathbf{q},\mathbf{p}} f(\overline{\mathbf{D}}_{\mathbf{p}} g)$ provided that f is pure of grade $\mathbf{q}$

(3.11) $(fg)\overleftarrow{\mathbf{D}}_{\mathbf{p}} = \delta^{\mathbf{p},\mathbf{q}}(f\overleftarrow{\mathbf{D}}_{\mathbf{p}})g + f(g\overleftarrow{\mathbf{D}}_{\mathbf{p}})$ provided that g is pure of grade $\mathbf{q}$

Proof. As in Lavendhomme (1996, §1.2, Proposition 1). ■

Now we discuss a simple variant of Taylor's formula for a function f: $\mathbb{R}_{\mathbf{p}_1} \times \cdots \times \mathbb{R}_{\mathbf{p}_n} \to \mathbb{R}$. We denote by $\overleftarrow{\partial}/\partial x_i$ the operator $\overleftarrow{\mathrm{D}}_{\mathbf{p}_i}$ $(1 \leq i < n)$. The formula goes as follows:

Theorem 3.2. Let $\underline{a} \in \mathbb{R}_{\mathbf{p}_1} \times \cdots \times \mathbb{R}_{\mathbf{p}_n}$. Then there exist unique $b_{k,i_1 \ldots i_k} \in \mathbb{R}$ for each k $(0 \leq k \leq n)$ and each sequence $1 \leq i_1 < \cdots < i_k \leq n$ such that for any $\underline{d} = (d_1, \ldots, d_n) \in D(\mathbf{p}_1) \times \cdots \times D(\mathbf{p}_n)$,

$$(3.12)\quad f(\underline{a} + \underline{d}) = a_0 + \sum_{i=1}^{n} b_{1,i} d_i + \sum_{i_1 < i_2} b_{2,i_1 i_2} d_{i_1} d_{i_2} + \cdots + \sum_{1 \leq i_1 < \cdots < i_k \leq n} b_{k,i_1 \ldots i_k} d_{i_1} \ldots d_{i_k} + \cdots + b_{n,1\ldots n} d_1 \ldots d_n$$

More specifically, we have

$$(3.13)\quad b_{k,i_1 \ldots i_k} = \left(f \frac{\overleftarrow{\partial}}{\partial x_k} \cdots \frac{\overleftarrow{\partial}}{\partial x_i} \right)(\underline{a})$$

Proof. As in Lavendhomme (1996, §§1.2.2). ■

4. BRAIDED TANGENCY

Let $\mathcal{M}$ be a microlinear space and $\mathrm{m}_0 \in \mathcal{M}$. These entities shall be fixed throughout this and the next sections. A *vector tangent to* $\mathcal{M}$ *at* m_0 is a mapping t: $D(\mathbf{1}, \ldots, \mathbf{k}) \to \mathcal{M}$ with $t(0, \ldots, 0) = \mathrm{m}_0$. Now we would like to endow the set $\mathrm{T}_{\mathrm{m}_0}\mathcal{M}$ of tangent vectors to $\mathcal{M}$ at m_0 with an $\mathbb{R}$-module structure. The set $\mathrm{T}_{\mathrm{m}_0}\mathcal{M}$ is called the *braided tangent space of* $\mathcal{M}$ *at* m_0. The left product $a \cdot t$ of $t \in \mathrm{T}_{\mathrm{m}_0}\mathcal{M}$ by $a \in \mathbb{R}$ and the right product $t \cdot b$ of t by $b \in \mathbb{R}$ are defined by the following formulas:

(4.1) $(a \cdot t)(d) = t(da)$

(4.2) $(t \cdot b)(d) = t(bd)$

for any $d \in D(\mathbf{1}, \ldots, \mathbf{k})$. Given $t_1, t_2 \in \mathrm{T}_{\mathrm{m}_0}\mathcal{M}$, their sum $t_1 + t_2$ is defined to be

(4.3) $(t_1 + t_2)(d) = l^{(\mathbf{1},\ldots,\mathbf{k})^2}_{(t_1,t_2)}(d, d)$

for any $d \in D(\mathbf{1}, \ldots, \mathbf{k})$.

Proposition 4.1. With the above operations the set $\mathrm{T}_{\mathrm{m}_0}\mathcal{M}$ is an $\mathbb{R}$-$\mathfrak{C}$-bimodule.

Proof. As in Lavendhomme (1996, §3.1, Proposition 1). ■

Proposition 4.2. The $\mathbb{R}$-$\mathfrak{C}$-bimodule $\mathrm{T}_{\mathrm{m}_0}\mathcal{M}$ is Euclidean in the sense that it satisfies the following condition:

(4.4) For any function f: $D(\mathbf{p}) \to \mathrm{T}_{\mathrm{m}_0}\mathcal{M}$, there exists a unique $t \in \mathrm{T}_{\mathrm{m}_0}\mathcal{M}$ such that $f(d) = f(0) + d \cdot t$ for any $d \in D(\mathbf{p})$.

Proof. As in Lavendhomme (1996, §§3.1, Proposition 3.2). ■

Now we define *pure tangent spaces* $\mathrm{T}^{\mathbf{p}}_{\mathrm{m}_0}\mathcal{M}$ *of* $\mathcal{M}$ *at* m_0 to be the set of functions t: $D(\mathbf{p}) \to \mathcal{M}$ with $t(0) = \mathrm{m}_0$. It is endowed with a k-linear space structure by decreeing that for any $a \in \mathrm{k}$, any t, t_1, $t_2 \in \mathrm{T}^{\mathbf{p}}_{\mathrm{m}_0}\mathcal{M}$ and any $d \in D(\mathbf{p})$,

(4.5) $(t_1 + t_2)(d) = l^{\mathbf{p}^2}_{(t_1,t_2)}(d, d)$
(4.6) $(a \cdot t)(d) = t(ad)$

Proposition 4.3. With the above operation the set $\mathrm{T}^{\mathbf{p}}_{\mathrm{m}_0}\mathcal{M}$ is a k-linear space.

Proof. As in Lavendhomme (1996, §3.1, Proposition 1). ■

The injections $i^{\mathbf{1},\ldots,\mathbf{k}}_{\mathbf{p}}$: $D(\mathbf{p}) \to D(\mathbf{1}, \ldots, \mathbf{k})$ induce functions $\mathfrak{p}_{\mathbf{p}}$: $\mathrm{T}_{\mathrm{m}_0}\mathcal{M} \to \mathrm{T}^{\mathbf{p}}_{\mathrm{m}_0}\mathcal{M}$. Similarly the projections $\mathrm{p}^{\mathbf{1},\ldots,\mathbf{k}}_{\mathbf{p}}$: $D(\mathbf{1}, \ldots, \mathbf{k}) \to D(\mathbf{p})$ induce functions $\mathfrak{i}_{\mathbf{p}}$: $\mathrm{T}^{\mathbf{p}}_{\mathrm{m}_0}\mathcal{M} \to \mathrm{T}_{\mathrm{m}_0}\mathcal{M}$. Then we have the following result.

Lemma 4.4. $\mathrm{T}_{\mathrm{m}_0}\mathcal{M}$ is a biproduct of $\mathrm{T}^{\mathbf{p}}_{\mathrm{m}_0}\mathcal{M}$'s within the abelian category of k-linear spaces in the sense that

(4.7) $\mathfrak{p}_{\mathbf{p}} \circ \mathfrak{i}_{\mathbf{p}} = 1_{\mathrm{T}^{\mathbf{p}}_{\mathrm{m}_0}\mathcal{M}}$ for any $\mathbf{p}$ in Π
(4.8) $\mathfrak{i}_{\mathbf{1}} \circ \mathfrak{p}_{\mathbf{1}} + \cdots + \mathfrak{i}_{\mathbf{k}} \circ \mathfrak{p}_{\mathbf{k}} = 1_{\mathrm{T}_{\mathrm{m}_0}\mathcal{M}}$

Proof. As in Nishimura (1998b, Lemma 4.5). ■

If $\mathrm{T}^{\mathbf{p}}_{\mathrm{m}_0}\mathcal{M}$ is to be regarded as k-linear subspaces of $\mathrm{T}_{\mathrm{m}_0}\mathcal{M}$ in the above sense, then it is not difficult to see that $\mathrm{T}^{\mathbf{p}}_{\mathrm{m}_0}$ is exactly the p-component of $\mathrm{T}_{\mathrm{m}_0}\mathcal{M}$. If $\mathcal{M}$ is $\mathbb{R}_{\mathbf{p}}$ and Π is not only a monoid but a group, then the $\mathbb{R}$-$\mathfrak{C}$-module $\mathrm{T}_{\mathrm{m}_0}\mathcal{M}$ is easily seen to be canonically isomorphic to $\mathbb{R}$, where $1 \in \mathbb{R}$ corresponds to the pure tangent vector $d \in D(\mathbf{p}) \mapsto \mathrm{m}_0 + \mathrm{d}$. We set $\mathrm{T}^{\mathbf{p}}\mathcal{M} = \bigcup_{\mathrm{m}\in\mathcal{M}}\mathrm{T}^{\mathbf{p}}_{\mathrm{m}}\mathcal{M}$.

A *vector field on* $\mathcal{M}$ is a tangent vector to $\mathcal{M}^{\mathcal{M}}$ at $1_{\mathcal{M}}$, i.e., it is an assignment X of an infinitesimal transformation X_d: $\mathcal{M} \to \mathcal{M}$ to each $d \in$

$D(\mathbf{1}, \ldots, \mathbf{k})$ with $X_0 = 1_{\mathcal{M}}$. The totality of vector fields on $\mathcal{M}$ is denoted by $\chi(\mathcal{M})$. As we discussed in Lemma 4.4, the $\mathbb{R}$-module $\chi(\mathcal{M})$ can be decomposed into its pure parts $\chi^{\mathbf{p}}(\mathcal{M})$, which consists of all assignments X of an infinitesimal transformation X_d: $\mathcal{M} \to \mathcal{M}$ to each $d \in D(\mathbf{p})$ with $X_0 = 1_{\mathcal{M}}$.

Given two pure vector fields X, Y on $\mathcal{M}$, we now define their Lie bracket $[X, Y]$ by Corollary 2.3 as follows:

(4.9) If $X \in \chi^{\mathbf{p}}(M)$ and $Y \in \chi^{\mathbf{q}}(M)$, then $[X, Y]$ is the unique vector field of type $\mathbf{p} + \mathbf{q}$ on $\mathcal{M}$ such that $[X, Y]_{d_1 d_2} = Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1}$ for any $d_1 \in D(\mathbf{p})$ and any $d_2 \in D(\mathbf{q})$.

Once the Lie bracket of any two pure vector fields on $\mathcal{M}$ is defined, we can define the Lie bracket $[X, Y]$ of two nonpure vector fields X, Y on $\mathcal{M}$ by the following formula:

(4.10) $[X, Y] = \Sigma_{\mathbf{p},\mathbf{q} \in \Pi}[X_{\mathrm{p}}, Y_{\mathrm{q}}]$

The proof of the following theorem is relegated to the succeeding section.

Theorem 4.5. $\chi(M)$ is a Lie $\mathbb{R}$-$\mathfrak{C}$-algebra.

5. MICROSQUARES AND MICROCUBES

The main objective of this section is to discuss fundamental properties of microsquares and microcubes in our braided context and apply them to Lie brackets of vector fields.

A *microsquare of type* $(\mathbf{p}, \mathbf{q})$ *on* $\mathcal{M}$ *at* $\mathrm{m} \in \mathcal{M}$ is a function α from $D(\mathbf{p}) \times D(\mathbf{q})$ to $\mathcal{M}$ with $\alpha(0, 0) = \mathrm{m}$. The totality of microsquares of type $(\mathbf{p}, \mathbf{q})$ on $\mathcal{M}$ at m is denoted by $\mathrm{T}^{\mathbf{p},\mathbf{q}}_{\mathrm{m}}\mathcal{M}$, and we set $\mathrm{T}^{\mathbf{p},\mathbf{q}}\mathcal{M} = \cup_{\mathrm{m} \in \mathcal{M}} \mathrm{T}^{\mathbf{p},\mathbf{q}}_{\mathrm{m}}\mathcal{M}$.

Lemma 5.1. The diagram

$$\begin{array}{ccc} D(\mathbf{p}, \mathbf{q}) & \xrightarrow{\;i\;} & D(\mathbf{p}) \times D(\mathbf{q}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \psi^{\mathbf{p},\mathbf{q}}} \\ D(\mathbf{p}) \times D(\mathbf{q}) & \xrightarrow[\varphi^{\mathbf{p},\mathbf{q}}]{} & (D(\mathbf{p}) \times D(\mathbf{q})) \vee D(\mathbf{p} + \mathbf{q}) \end{array}$$

is a quasi-colimit diagram of small objects, where

(5.1) $(D(\mathbf{p}) \times D(\mathbf{q})) \vee D(\mathbf{p} + \mathbf{q})$
$= \{(d_1, d_2, d_3) \in D(\mathbf{p}) \times D(\mathbf{q}) \times D(\mathbf{p} + \mathbf{q}) \mid d_1 d_3 = d_2 d_3 = 0\}$

(5.2) $\varphi^{\mathbf{p},\mathbf{q}}(d_1, d_2) = (d_1, d_2, 0)$ for any $(d_1, d_2) \in D(\mathbf{p}) \times D(\mathbf{q})$

(5.3) $\psi^{\mathbf{p},\mathbf{q}}(d_1, d_2) = (d_1, d_2, d_1 d_2)$ for any $(d_1, d_2) \in D(\mathbf{p}) \times D(\mathbf{q})$

Proof. As in Lavendhomme (1996, §3.4, pp. 92–93, Lemma). ■

Proposition 5.2. For any $\alpha_1, \alpha_2 \in \mathrm{T}^{\mathbf{p},\mathbf{q}}\mathcal{M}$, if $\alpha_1|_{D(\mathbf{p},\mathbf{q})} = \alpha_2|_{D(\mathbf{p},\mathbf{q})}$, then there exists a unique function $\mathtt{g}^{\mathbf{p},\mathbf{q}}_{(\alpha_1,\alpha_2)}: (D(\mathbf{p}) \times D(\mathbf{q})) \vee D(\mathbf{p}+\mathbf{q}) \to \mathcal{M}$ such that $\mathtt{g}^{\mathbf{p},\mathbf{q}}_{(\alpha_1,\alpha_2)} \circ \varphi^{\mathbf{p},\mathbf{q}} = \alpha_1$ and $\mathtt{g}^{\mathbf{p},\mathbf{q}}_{(\alpha_1,\alpha_2)} \circ \psi^{\mathbf{p},\mathbf{q}} = \alpha_2$. In this case we define a pure tangent vector $\alpha_2 \underset{\mathbf{p},\,\mathbf{q}}{\dot{-}} \alpha_1$ of type $\mathbf{p}+\mathbf{q}$ to $\mathcal{M}$ as follows:

$$(5.4) \quad \left(\alpha_2 \underset{\mathbf{p},\,\mathbf{q}}{\dot{-}} \alpha_1\right)(d) = \mathtt{g}^{\mathbf{p},\mathbf{q}}_{(\alpha_1,\alpha_2)}(0, 0, d) \quad \text{for any} \quad d \in D(\mathbf{p}+\mathbf{q})$$

Proof. This follows from Lemma 5.1. ■

Proposition 5.3. For any $\alpha_1, \alpha_2 \in \mathrm{T}^{\mathbf{p},\mathbf{q}}_m\mathcal{M}$ with $\alpha_1|_{D(\mathbf{p},\mathbf{q})} = \alpha_2|_{D(\mathbf{p},\mathbf{q})}$, we have

$$(5.5) \quad \alpha_1 \underset{\mathbf{p},\,\mathbf{q}}{\dot{-}} \alpha_2 = -\left(\alpha_2 \underset{\mathbf{p},\,\mathbf{q}}{\dot{-}} \alpha_1\right)$$

Proof. We define $\mathtt{h}: (D(\mathbf{p}) \times D(\mathbf{q})) \vee D(\mathbf{p}+\mathbf{q}) \to \mathcal{M}$ as follows:

$$(5.6) \quad \mathtt{h}(d_1, d_2, d_3) = \mathtt{g}^{\mathbf{p},\mathbf{q}}_{(\alpha_1,\alpha_2)}(d_1, d_2, d_1d_2 - d_3) \text{ for any } (d_1, d_2, d_3) \in (D(\mathbf{p}) \times D(\mathbf{q})) \vee D(\mathbf{p}+\mathbf{q})$$

Then it is easy to see that $\mathtt{h} \circ \varphi^{\mathbf{p},\mathbf{q}} = \alpha_2$ and $\mathtt{h} \circ \psi^{\mathbf{p},\mathbf{q}} = \alpha_1$. Therefore $\mathtt{h} = \mathtt{g}^{\mathbf{p},\mathbf{q}}_{(\alpha_2,\alpha_1)}$, which implies (5.5) at once. ■

For any $\alpha \in \mathrm{T}^{\mathbf{p},\mathbf{q}}\mathcal{M}$, we define $\Sigma(\alpha) \in \mathrm{T}^{\mathbf{q},\mathbf{p}}\mathcal{M}$ to be

$$(5.7) \quad \Sigma(\alpha)(d_1, d_2) = \alpha(d_2, d_1) \quad \text{for any} \quad (d_1, d_2) \in D(\mathbf{p}) \times D(\mathbf{q})$$

The following proposition reveals the underlying structure of the braided anticommutativity of vector fields with respect to Lie brackets.

Proposition 5.4. For any $\alpha_1, \alpha_2 \in \mathrm{T}^{\mathbf{p},\mathbf{q}}\mathcal{M}$ with $\alpha_1|_{D(\mathbf{p},\mathbf{q})} = \alpha_2|_{D(\mathbf{p},\mathbf{q})}$, we have

$$(5.8) \quad \Sigma(\alpha_1)|_{D(\mathbf{q},\mathbf{p})} = \Sigma(\alpha_2)|_{D(\mathbf{q},\mathbf{p})}$$

$$(5.9) \quad \Sigma(\alpha_2) \underset{\mathbf{q},\,\mathbf{p}}{\dot{-}} \Sigma(\alpha_1) = \delta^{\mathbf{p},\mathbf{q}}\left(\alpha_2 \underset{\mathbf{p},\,\mathbf{q}}{\dot{-}} \alpha_1\right)$$

Proof. Let us define $\mathtt{h}: (D(\mathbf{q}) \times D(\mathbf{p})) \vee D(\mathbf{p}+\mathbf{q}) \to \mathcal{M}$ as follows:

$$(5.10) \quad \mathtt{h}(d_1, d_2, d_3) = \mathtt{g}^{\mathbf{p},\mathbf{q}}_{(\alpha_1,\alpha_2)}(d_2, d_1, \delta^{\mathbf{p},\mathbf{q}}d_3) \text{ for any } (d_1, d_2, d_3) \in (D(\mathbf{q}) \times D(\mathbf{p})) \vee D(\mathbf{p}+\mathbf{q})$$

Then it is easy to see that $\mathtt{h} \circ \varphi^{\mathbf{q},\mathbf{p}} = \Sigma(\alpha_1)$ and $\mathtt{h} \circ \psi^{\mathbf{q},\mathbf{p}} = \Sigma(\alpha_2)$, whence (5.9) follows. ■

Now we discuss a braided version of a microcube. A *microcube of type* $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ on $\mathcal{M}$ at $\mathtt{m} \in \mathcal{M}$ is a function γ from $D(\mathbf{p}) \times D(\mathbf{q}) \times D(\mathbf{r})$ to $\mathcal{M}$

with $\alpha(0, 0, 0) = \mathrm{m}$. The totality of microcubes of type $(\mathbf{p}, \mathbf{q}, \mathbf{r})$ on $\mathcal{M}$ at m is denoted by $\mathrm{T}_{\mathrm{m}}^{\mathbf{p},\mathbf{q},\mathbf{r}}\mathcal{M}$, and we set $\mathrm{T}^{\mathbf{p},\mathbf{q},\mathbf{r}}\mathcal{M} = \cup_{\mathrm{m}\in\mathcal{M}}\mathrm{T}_{\mathrm{m}}^{\mathbf{p},\mathbf{q},\mathbf{r}}\mathcal{M}$.

Now we relativize the partial binary operation $\underset{\mathbf{q},\,\mathbf{r}}{\overset{\cdot}{\longrightarrow}}$ to $\mathrm{T}^{\mathbf{p},\mathbf{q},\mathbf{r}}\mathcal{M}$. As discussed in Nishimura (1998a, Section 1.3), we can do so by regarding $\mathrm{T}^{\mathbf{p},\mathbf{q},\mathbf{r}}\mathcal{M}$ either as $\mathrm{T}^{\mathbf{p}}(\mathrm{T}^{\mathbf{q},\mathbf{r}}\mathcal{M})$ or as $\mathrm{T}^{\mathbf{q},\mathbf{r}}(\mathrm{T}^{\mathbf{p}}\mathcal{M})$. Fortunately both approaches result in the same partial operation $\underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{\mathrm{i}}{\longrightarrow}}$; given $\gamma_1, \gamma_2 \in \mathrm{T}^{\mathbf{p},\mathbf{q},\mathbf{r}}\mathcal{M}$, $\gamma_2 \underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{\mathrm{i}}{\longrightarrow}} \gamma_1$ is defined iff $\gamma_1|_{D(\mathbf{p})\times D(\mathbf{q},\mathbf{r})} = \gamma_2|_{D(\mathbf{p})\times D(\mathbf{q},\mathbf{r})}$, in which it is a microsquare of type $(\mathbf{p}, \mathbf{q} + \mathbf{r})$ on $\mathcal{M}$.

Let $\mathfrak{Perm}_3$ denote the group of permutations of the set $\{1, 2, 3\}$. Given $\gamma \in \mathrm{T}^{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3}M$ and $\rho \in \mathfrak{Perm}_3$, we define $\Sigma_\rho(\gamma) \in \mathrm{T}^{\mathbf{p}_{\rho^{-1}(1)},\mathbf{p}_{\rho^{-1}(2)},\mathbf{p}_{\rho^{-1}(3)}}\mathcal{M}$ as follows:

$$(5.11)\quad \Sigma_\rho(\gamma)(d_1, d_2, d_3) = \gamma(d_{\rho(1)}, d_{\rho(2)}, d_{\rho(3)}) \text{ for any } (d_1, d_2, d_3) \in D^{\mathbf{p}_{\rho^{-1}(1)}} \times D^{\mathbf{p}_{\rho^{-1}(2)}} \times D^{\mathbf{p}_{\rho^{-1}(3)}}$$

Now we define partial binary operations $\underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{2}{\longrightarrow}}$ and $\underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{3}{\longrightarrow}}$ in $\mathrm{T}^{\mathbf{p},\mathbf{q},\mathbf{r}}\mathcal{M}$ as follows:

(5.12) $\gamma_2 \underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{2}{\longrightarrow}} \gamma_1$ is defined iff
$\Sigma_{(132)}(\gamma_2) \underset{\mathbf{q},\,\mathbf{r},\,\mathbf{p}}{\overset{\mathrm{i}}{\longrightarrow}} \Sigma_{(132)}(\gamma_1)$ is defined, in which the former is defined to be the latter.

(5.13) $\gamma_2 \underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{3}{\longrightarrow}} \gamma_1$ is defined iff
$\Sigma_{(123)}(\gamma_2) \underset{\mathbf{r},\,\mathbf{p},\,\mathbf{q}}{\overset{\mathrm{i}}{\longrightarrow}} \Sigma_{(123)}(\gamma_1)$ is defined, in which the former is defined to be the latter.

The following theorem reveals the underlying structure of the braided Jacobi identity of Lie brackets of vector fields.

Theorem 5.5. Let $\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321} \in \mathrm{T}_{\mathrm{m}}^{\mathbf{p},\mathbf{q},\mathbf{r}}\mathcal{M}$. Let us suppose that the following three expressions are well defined:

$$(5.14)\quad \left(\gamma_{123} \underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{\mathrm{i}}{\longrightarrow}} \gamma_{132}\right) \underset{\mathbf{p},\,\mathbf{q}+\mathbf{r}}{\overset{\cdot}{\longrightarrow}} \left(\gamma_{231} \underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{\mathrm{i}}{\longrightarrow}} \gamma_{321}\right)$$

$$(5.15)\quad \left(\gamma_{231} \underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{2}{\longrightarrow}} \gamma_{213}\right) \underset{\mathbf{q},\,\mathbf{p}+\mathbf{r}}{\overset{\cdot}{\longrightarrow}} \left(\gamma_{312} \underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{2}{\longrightarrow}} \gamma_{132}\right)$$

$$(5.16)\quad \left(\gamma_{312} \underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{3}{\longrightarrow}} \gamma_{321}\right) \underset{\mathbf{r},\,\mathbf{p}+\mathbf{q}}{\overset{\cdot}{\longrightarrow}} \left(\gamma_{123} \underset{\mathbf{p},\,\mathbf{q},\,\mathbf{r}}{\overset{3}{\longrightarrow}} \gamma_{213}\right)$$

Letting ξ_1, ξ_2, and ξ_3 denote the above three expressions in order, we have

(5.17) $\xi_1 + \delta^{\mathbf{p},\mathbf{q}+\mathbf{r}}\xi_2 + \delta^{\mathbf{p}+\mathbf{q},\mathbf{r}}\xi_3 = 0$

Proof. As in Nishimura (1997, §3). ■

Now we apply the above theory of microsquares and microcubes to Lie brackets of vector fields. We denote by $\chi^{\mathbf{p},\mathbf{q}}(\mathcal{M})$ the totality of microsquares on $\mathcal{M}^{\mathcal{M}}$ at $1_{\mathcal{M}}$. We denote by $\chi^{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathcal{M})$ the totality of microcubes on $\mathcal{M}^{\mathcal{M}}$ at $1_{\mathcal{M}}$. Given $X \in \chi^{\mathbf{p}}(\mathcal{M})$, $Y \in \chi^{\mathbf{q}}(\mathcal{M})$, and $Z \in \chi^{\mathbf{r}}(\mathcal{M})$, we define $Y * X \in \chi^{\mathbf{p},\mathbf{q}}(\mathcal{M})$ and $Z * Y * X \in \chi^{\mathbf{p},\mathbf{q},\mathbf{r}}(\mathcal{M})$ as follows:

(5.18) $(Y * X)(d_1, d_2) = Y_{d_2} \circ X_{d_1}$ for any $(d_1, d_2) \in D(\mathbf{p}) \times D(\mathbf{q})$

(5.19) $(Z * Y * X)(d_1, d_2, d_3) = Z_{d_3} \circ Y_{d_2} \circ X_{d_1}$ for any $(d_1, d_2, d_3) \in D(\mathbf{p}) \times D(\mathbf{q}) \times D(\mathbf{r})$

Proposition 5.6. Let $X \in \chi^{\mathbf{p}}(\mathcal{M})$ and $Y \in \chi^{\mathbf{q}}(\mathcal{M})$. Then we have

(5.20) $[X, Y] = Y * X \underset{\mathbf{p},\,\mathbf{q}}{\overset{\cdot}{-}} \Sigma(X * Y)$

Proof. As in Lavendhomme (1996, §3.4, Proposition 8). ■

Theorem 5.7. Let $X \in \chi^{\mathbf{p}}(\mathcal{M})$ and $Y \in \chi^{\mathbf{q}}(\mathcal{M})$. Then we have

(5.21) $[X, Y] = -\delta^{\mathbf{p},\mathbf{q}}[Y, X]$

Proof. We have

$$
\begin{aligned}
&[X, Y] \\
&= Y * X \underset{\mathbf{p},\,\mathbf{q}}{\overset{\cdot}{-}} \Sigma(X * Y) \\
&= -\left(\Sigma(X * Y) \underset{\mathbf{p},\,\mathbf{q}}{\overset{\cdot}{-}} Y * X\right) \quad \text{[Proposition 5.3]} \\
&= -\delta^{\mathbf{p},\mathbf{q}}\left(X * Y \underset{\mathbf{q},\,\mathbf{p}}{\overset{\cdot}{-}} \Sigma(Y * X)\right) \quad \text{[Proposition 5.4]} \\
&= -\delta^{\mathbf{p},\mathbf{q}}[Y, X]
\end{aligned}
$$
■

Proposition 5.8. Let $X \in \chi^{\mathbf{p}}(\mathcal{M})$, $Y \in \chi^{\mathbf{q}}(\mathcal{M})$, and $Z \in \chi^{\mathbf{r}}(\mathcal{M})$. Let it be the case that

(5.22) $\gamma_{123} = Z * Y * X$

(5.23) $\gamma_{132} = \Sigma_{(23)}(Y * Z * X)$

(5.24) $\gamma_{213} = \Sigma_{(12)}(Z * X * Y)$

(5.25) $\gamma_{231} = \Sigma_{(123)}(X * Z * Y)$

$$(5.26)\quad \gamma_{312} = \Sigma_{(132)}(Y * X * Z)$$
$$(5.27)\quad \gamma_{321} = \Sigma_{(13)}(X * Y * Z)$$

Then the right-hand sides of the following three identities are meaningful, and all the three identities hold:

$$(5.28)\quad [X, [Y, Z]] = \left(\gamma_{123} \frac{\mathrm{i}}{\mathbf{p}, \mathbf{q}, \mathbf{r}} \gamma_{132}\right) \frac{\cdot}{\mathbf{p}, \mathbf{q} + \mathbf{r}} \left(\gamma_{231} \frac{\mathrm{i}}{\mathbf{p}, \mathbf{q}, \mathbf{r}} \gamma_{321}\right)$$

$$(5.29)\quad [Y, [Z, X]] = \left(\gamma_{231} \frac{2}{\mathbf{p}, \mathbf{q}, \mathbf{r}} \gamma_{213}\right) \frac{\cdot}{\mathbf{q}, \mathbf{p} + \mathbf{r}} \left(\gamma_{312} \frac{2}{\mathbf{p}, \mathbf{q}, \mathbf{r}} \gamma_{132}\right)$$

$$(5.30)\quad [Z, [X, Y]] = \left(\gamma_{312} \frac{3}{\mathbf{p}, \mathbf{q}, \mathbf{r}} \gamma_{321}\right) \frac{\cdot}{\mathbf{r}, \mathbf{p} + \mathbf{q}} \left(\gamma_{123} \frac{3}{\mathbf{p}, \mathbf{q}, \mathbf{r}} \gamma_{213}\right)$$

Proof. As in Nishimura (1997a, Proposition 2.7). ■

Theorem 5.9. Let $X \in \chi^{\mathbf{p}}(\mathcal{M})$, $Y \in \chi^{\mathbf{q}}(\mathcal{M})$, and $Z \in \chi^{\mathbf{r}}(\mathcal{M})$. Then

$$(5.31)\quad [X, [Y, Z]] + \delta^{\mathbf{p}, \mathbf{q}+\mathbf{r}}[Y, [Z, X]] + \delta^{\mathbf{p}+\mathbf{q}, \mathbf{r}}[Z, [X, Y]] = 0$$

Proof. Follows from Theorem 5.5 and Proposition 5.8. ■

We conclude this section by remarking that Theorems 5.7 and 5.9 constitute a proof of Theorem 4.5.

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